Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

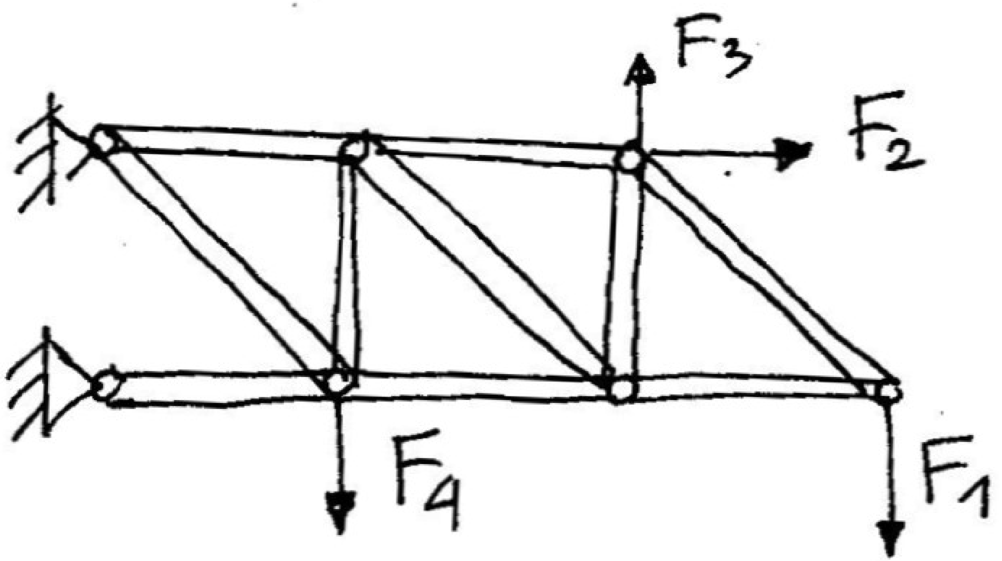
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

2D truss finite element

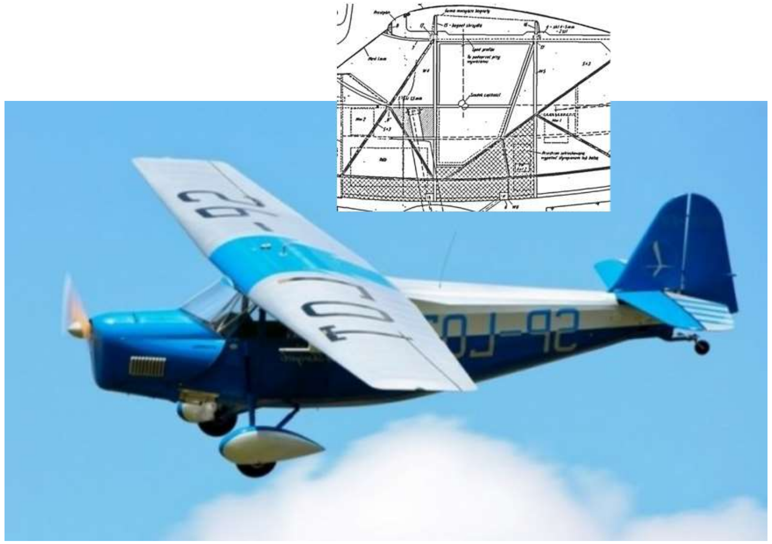
04.2021

Trusses - structures made of axial straight bars joined at nodes. Only tensile and compressive forces in members are considered - other internal forces are excluded because all joints are treated as articulated joints.



Examples of trusses

Bridge

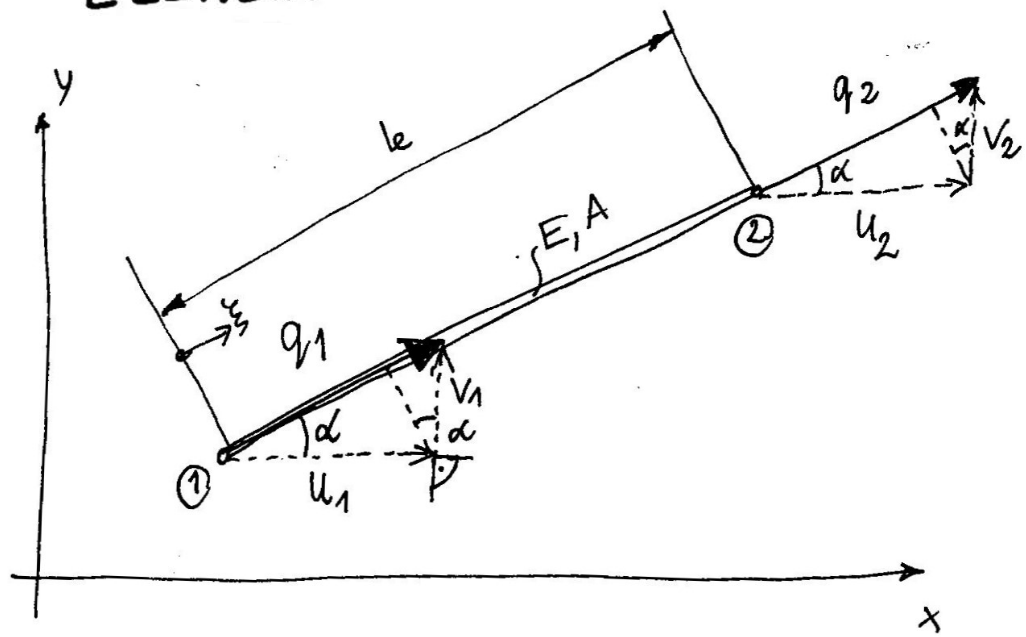


Fuselage

Tower crane

Roof truss

2D TRUSS ELEMENT



AXIAL BAR ELEMENT $\Longrightarrow$ 2D TRUSS ELEMENT

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e \Longrightarrow \{q_g\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}_e$$

$$c = \cos\alpha, \quad s = \sin\alpha$$

$$q_1 = c \cdot u_1 + s \cdot v_1 + 0 \cdot u_2 + 0 \cdot v_2$$

$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + c \cdot u_2 + s \cdot v_2$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}}_{[T_t]_e} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}_e$$

$[T_t]_e$ — transformation matrix

$$\underset{2\times 1}{\{q\}_e} = \underset{2\times 4}{[T_t]_e} \cdot \underset{4\times 1}{\{q_g\}_e}$$

$$\underset{1\times 2}{\lfloor q \rfloor_e} = \underset{1\times 4}{\lfloor q_g \rfloor_e} \cdot \underset{4\times 2}{[T_t]_e^T}$$

elastic strain energy.

$$U_e = \frac{1}{2} \lfloor q \rfloor_e \cdot [k]_e \cdot \{q\}_e = \frac{1}{2} \underset{1\times 4}{\lfloor q_g \rfloor_e} \cdot \underset{4\times 2}{[T_t]_e^T} \cdot \underset{2\times 2}{[k]_e} \cdot \underset{2\times 4}{[T_t]_e} \cdot \underset{4\times 1}{\{q_g\}_e} =$$

$$= \frac{1}{2} \underset{1\times 4}{\lfloor q_g \rfloor_e} \cdot \underset{4\times 4}{[k_g]_e} \cdot \underset{4\times 1}{\{q_g\}_e} \quad , \text{ where:}$$

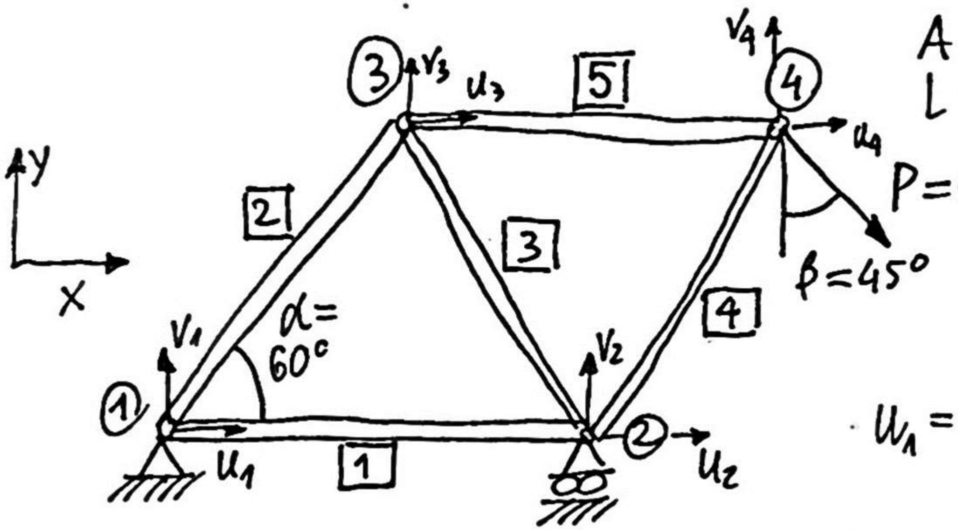
$$\underset{4\times 4}{[k_g]_e} = \frac{EA}{l_e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

EXAMPLE. BUILD A FE MODEL OF A 2-D TRUSS. FIND NODAL DISPLACEMENTS, STRESSES, INTERNAL FORCES AND REACTIONS

$E = 2 \cdot 10^5$ MPa

$A = 100$ mm^2

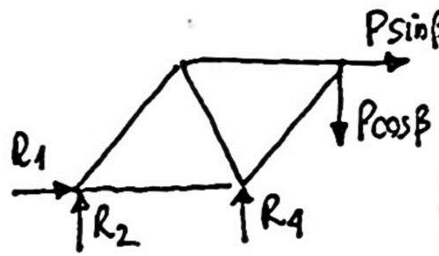
$L = 1000$ mm



$P = 450$ N

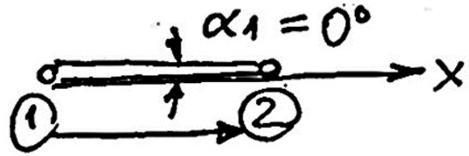
$u_1 = 0, \; v_1 = 0, \; v_2 = 0$

$$\underset{8\times1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix},$$



$$\underset{8\times1}{\{F\}} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ R_4 \\ 0 \\ 0 \\ P\sin\beta \\ -P\cos\beta \end{Bmatrix}$$

$\boxed{1}$

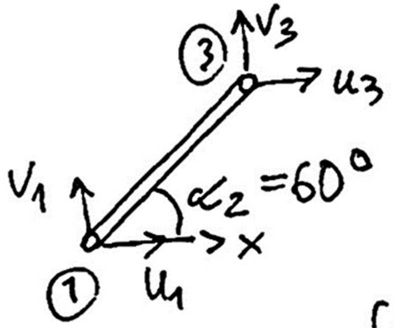


$c_1 = 1 \;,\; s_1 = 0 \;,\; \underset{1\times 4}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, u_2, v_2 \rfloor$

$$\underset{4\times 4}{\lfloor k_g \rceil_1} = \frac{EA}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{4l}\begin{bmatrix} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{8\times 8}{\lfloor k_g \rceil_1^*} = \frac{EA}{4l}\begin{bmatrix} 4 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{2}$

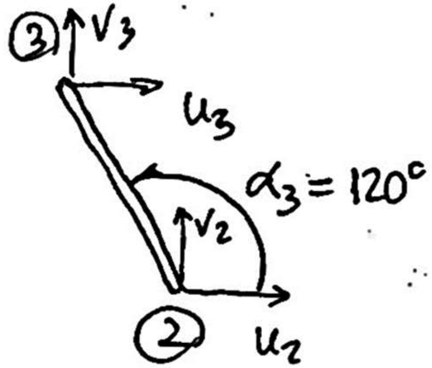


$c_2 = \frac{1}{2}$, $s_2 = \frac{\sqrt{3}}{2}$, $\underset{1\times 4}{\lfloor q_g \rfloor_2} = \lfloor u_1, v_1, u_3, v_3 \rfloor$

$$\underset{4\times 4}{[k_g]_2} = \frac{EA}{4l} \begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}$$

$$\underset{8\times 8}{[K_g]_2^*} = \frac{EA}{4l} \begin{bmatrix} 1 & \sqrt{3} & 0 & 0 & -1 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & -3 & 0 & 0 & \sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{3}$

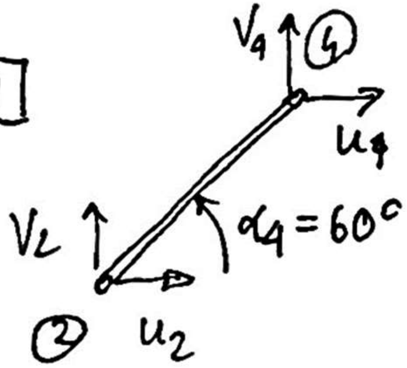


$$C_3 = -\frac{1}{2}, \quad S_3 = \frac{\sqrt{3}}{2}, \quad \lfloor q_g \rfloor_3 = \lfloor u_2, v_2, u_3, v_3 \rfloor$$

$$\underset{4\times4}{[k_g]_3} = \frac{EA}{4L}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}$$

$$\underset{8\times8}{[k_g]_3^*} = \frac{EA}{4L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} & -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 & 0 & 0 \\ 0 & 0 & -1 & \sqrt{3} & 1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & -3 & -\sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{4}$

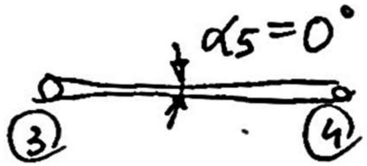


$c_4 = \frac{1}{2}, \quad s_4 = \frac{\sqrt{3}}{2}, \quad \lfloor q_g \rfloor_4 = \lfloor u_2, v_2, u_4, v_4 \rfloor$

$$\underset{4\times4}{[k_g]_4} = \frac{EA}{4l}\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}$$

$$\underset{8\times8}{[k_g]_4^*} = \frac{EA}{4l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} & 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & -3 & 0 & 0 & \sqrt{3} & 3 \end{bmatrix}$$

$\boxed{5}$



$C_5 = 1 \quad , S_5 = 0 \quad , \lfloor q_g \rfloor_5 = \lfloor u_3, v_3, u_4, v_4 \rfloor$

$$\underset{4\times 4}{\lfloor k_g \rfloor_5} = \frac{EA}{4l}\begin{bmatrix} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\underset{8\times 8}{\lfloor k_g \rfloor_5^{*}} = \frac{EA}{4l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_{8\times 8} = \sum_{e=1}^{5} [K_g]^*_e = \frac{EA}{4l}\begin{bmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 & 0 & 0 \\ -4 & 0 & 6 & 0 & -1 & \sqrt{3} & -1 & -\sqrt{3} \\ 0 & 0 & 0 & 6 & \sqrt{3} & -3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 6 & 0 & -4 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & -4 & 0 & 5 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & -3 & 0 & 0 & \sqrt{3} & 3 \end{bmatrix}$$

$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

\+ boundary conditions : $u_1 = 0$, $v_1 = 0$, $v_2 = 0$

$$[K]_{5\times5} \cdot \{q\}_{5\times1} = \{F\}_{5\times1}$$

$$\frac{EA}{4l}\begin{bmatrix} 6 & -1 & \sqrt{3} & -1 & -\sqrt{3} \\ -1 & 6 & 0 & -4 & 0 \\ \sqrt{3} & 0 & 6 & 0 & 0 \\ -1 & -4 & 0 & 5 & \sqrt{3} \\ -\sqrt{3} & 0 & 0 & \sqrt{3} & 3 \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2}P \\ -\frac{\sqrt{2}}{2}P \end{Bmatrix}$$

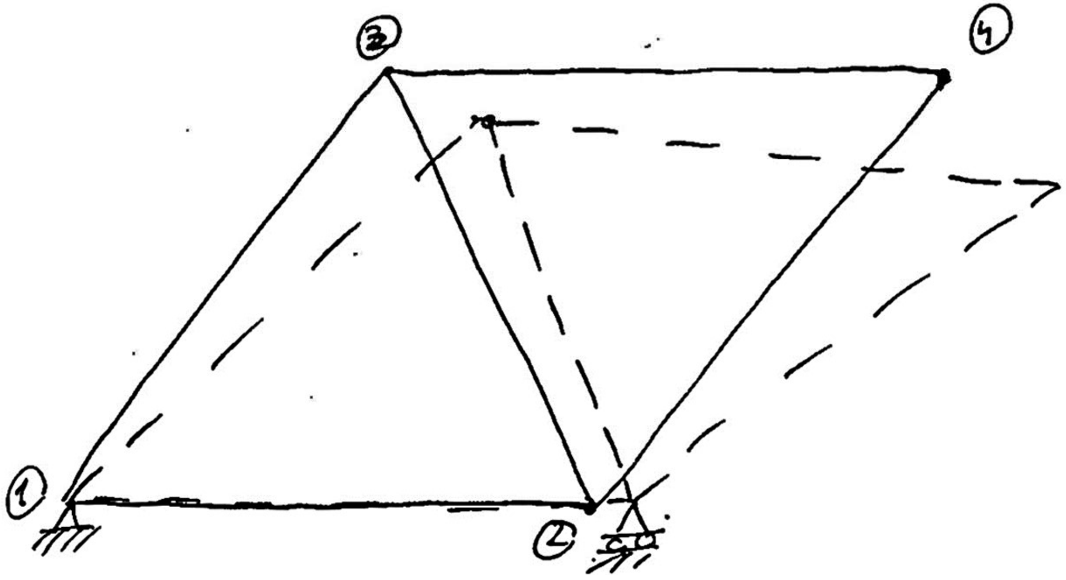
$$u_2 = 0.3362 \cdot 10^{-2}\,mm$$

$$u_3 = 5.1872 \cdot 10^{-2}\,mm$$

$$v_3 = -0.09706 \cdot 10^{-2}\,mm$$

$$u_4 = 7.6968 \cdot 10^{-2}\,mm$$

$$v_4 = -6.3705 \cdot 10^{-2}\,mm$$



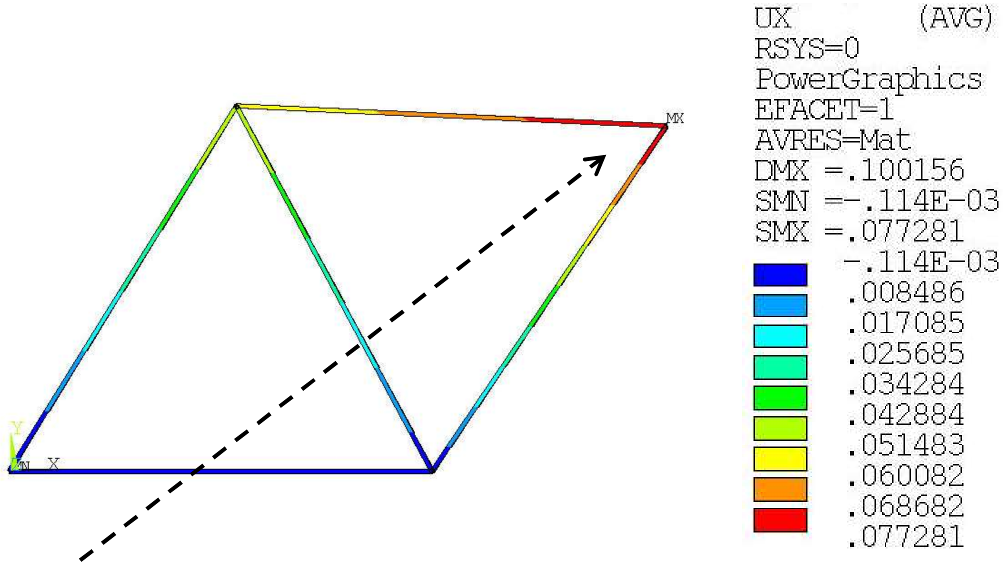
$$u_2 = 0.3362 \cdot 10^{-2}\ \text{mm}$$

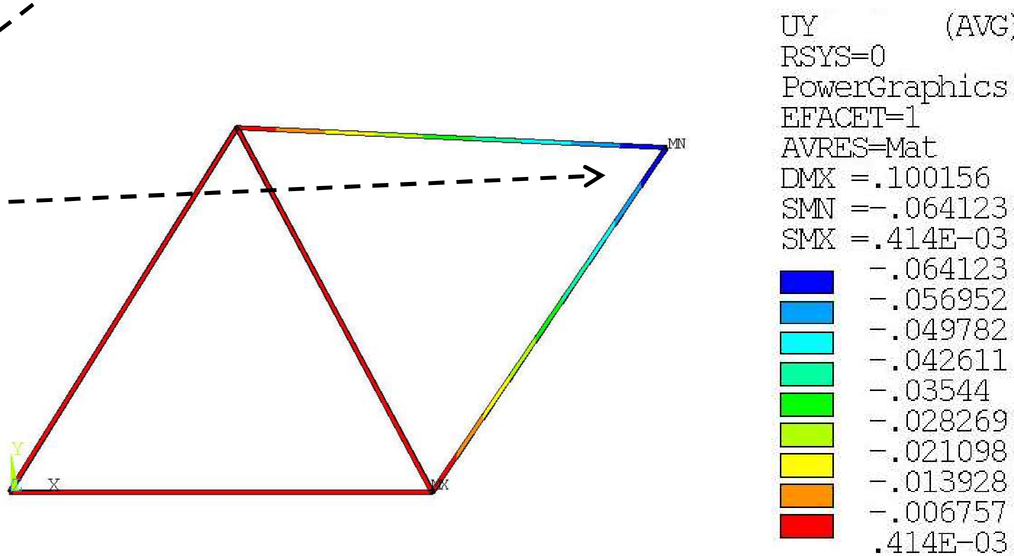
$$u_3 = 5.1872 \cdot 10^{-2}\ \text{mm}$$

$$v_3 = -0.09706 \cdot 10^{-2}\ \text{mm}$$

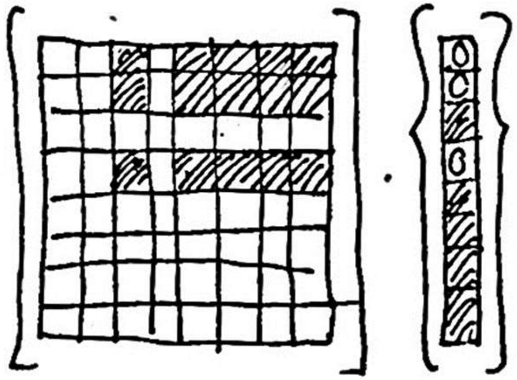
$$u_4 = 7.6968 \cdot 10^{-2}\ \text{mm}$$

$$v_4 = -6.3709 \cdot 10^{-2}\ \text{mm}$$

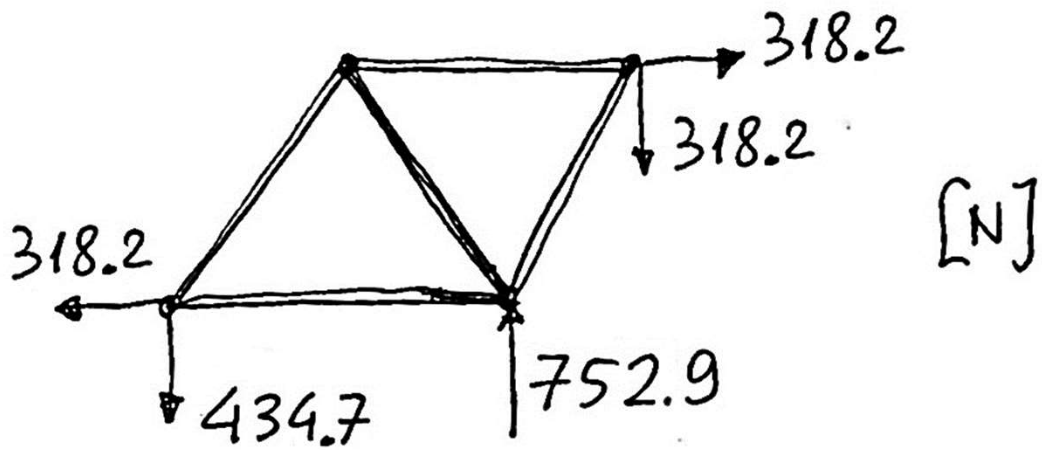




reactions



$$= \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ R_4 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2}P \\ -\frac{\sqrt{2}}{2}P \end{Bmatrix} \Rightarrow \begin{aligned} R_1 &= -318.2\ \text{N} \\ R_2 &= -434.7\ \text{N} \\ R_4 &= 752.9\ \text{N} \end{aligned}$$



stresses and internal forces :

$$\boxed{1} \quad \underset{2\times1}{\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1} = \underset{2\times4}{\begin{bmatrix} T_t \end{bmatrix}_1} \cdot \underset{4\times1}{\begin{Bmatrix} q_g \end{Bmatrix}_1} = \underset{(2\times4)}{\begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & C_1 & S_1 \end{bmatrix}} \cdot \underset{(4\times1)}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}_1} = \begin{Bmatrix} 0 \\ 0.33622\cdot10^{-2} \end{Bmatrix}$$

$$\sigma_1 = \frac{E}{l}\left(q_2 - q_1\right)_1 = 0.67\ \text{MPa} \quad , \quad N_1 = \sigma_1 \cdot A = 672.4\ \text{N}$$

$$\boxed{2} \quad \underset{2\times1}{\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_2} = \underset{2\times4}{\begin{bmatrix} T_t \end{bmatrix}_2} \cdot \underset{4\times1}{\begin{Bmatrix} q_g \end{Bmatrix}_2} = \begin{bmatrix} C_2 & S_2 & 0 & 0 \\ 0 & 0 & C_2 & S_2 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 2.5096\cdot10^{-2} \end{Bmatrix}$$

$$\sigma_2 = \frac{E}{l}\left(q_2 - q_1\right)_2 = 5.02\ \text{MPa} \quad , \quad N_2 = \sigma_2 \cdot A = 502\ \text{N}$$

$$\boxed{3} \quad \underset{2\times1}{\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_3} = \underset{2\times4}{\begin{bmatrix} T_t \end{bmatrix}_3} \cdot \underset{4\times1}{\begin{Bmatrix} q_g \end{Bmatrix}_3} = \begin{bmatrix} C_3 & S_3 & 0 & 0 \\ 0 & 0 & C_3 & S_3 \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_3 = \begin{Bmatrix} -0.16811\cdot10^{-2} \\ -2.67766\cdot10^{-2} \end{Bmatrix}$$

$$\sigma_3 = \frac{E}{l}\left(q_2 - q_1\right)_3 = -5.02\ \text{MPa} \quad , \quad N_3 = \sigma_3 \cdot A = -502\ \text{N}$$

(possible buckling)

$\boxed{4}$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_4 = [T_t]_4 \cdot \{q_g\}_4 = \begin{bmatrix} c_4 & s_4 & 0 & 0 \\ 0 & 0 & c_4 & s_4 \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0.16811 \cdot 10^{-2} \\ -1.66901 \cdot 10^{-2} \end{Bmatrix}$$

2×1

$$\sigma_4 = \frac{E}{L}(q_2 - q_1)_4 = -3.67\ \text{MPa}, \quad N_4 = \sigma_4 \cdot A = -367\ \text{N}$$

(possible buckling)

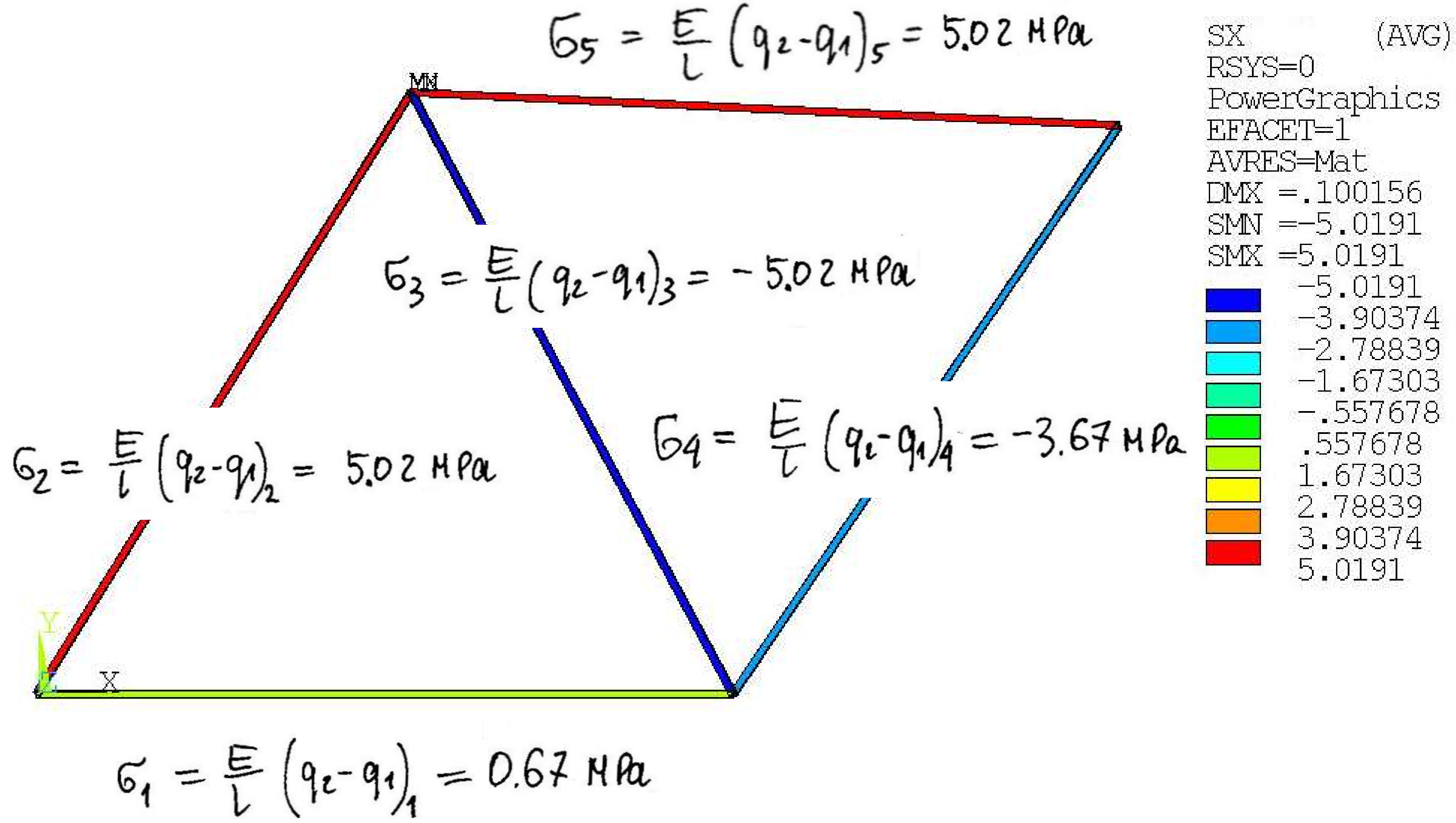
$\boxed{5}$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_5 = [T_t]_5 \cdot \{q_g\}_5 = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & c_5 & s_5 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 5.18721 \cdot 10^{-2} \\ 7.6968 \cdot 10^{-2} \end{Bmatrix}$$

$2 \times 1 \qquad 2 \times 4 \qquad 4 \times 1$

$$\sigma_5 = \frac{E}{L}(q_2 - q_1)_5 = 5.02\ \text{MPa}, \quad N_5 = 502\ \text{N}$$

Stress [MPa]



$\sigma_5 = \frac{E}{L}(q_2 - q_1)_5 = 5.02$ MPa
$\sigma_3 = \frac{E}{L}(q_2 - q_1)_3 = -5.02$ MPa
$\sigma_2 = \frac{E}{L}(q_2 - q_1)_2 = 5.02$ MPa
$\sigma_4 = \frac{E}{L}(q_2 - q_1)_4 = -3.67$ MPa
$\sigma_1 = \frac{E}{L}(q_2 - q_1)_1 = 0.67$ MPa
MN
MX
Y
X
SX (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.100156
SMN =-5.0191
SMX =5.0191
-5.0191
-3.90374
-2.78839
-1.67303
-.557678
.557678
1.67303
2.78839
3.90374
5.0191